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OPTIMAL DESIGN OF SOLID PLATES

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Abstract—The problem of optimal design of solid plates is concerned with the determination of a variable plate thickness that corresponds to minimum volume when the limit load is prescribed. In its solution, the condition of constancy of specific power of dissipation on the exterior plate surfaces is usually applied. This condition only assures a stationary value of the volume and may represent neither an absolute nor a local minimum. Examples of cantilever and circular plates are discussed, and alternative designs are presented which are of smaller volumes than the usual design. A modified formulation of the optimal design problem is discussed.

1. INTRODUCTION

CONSIDER a solid plate made of rigid, perfectly plastic material. For prescribed plan-form of the plate and conditions of loading and support, the problem of optimal design is concerned with the determination of a variable thickness $2h$ for which the plate volume attains a minimum when the limit load is prescribed. Thus

$$V = \int 2h\,\mathrm{d}A = \min. \tag{1.1}$$

Since the thickness can be expressed in terms of the limit bending moment $M_0 = \sigma_0 h^2$, we respectively obtain the following expressions for the Tresca and Mises yield conditions

$$V_T = \int [|M_1|+|M_2|+|M_1-M_2|]^{\frac{1}{2}}\,\mathrm{d}A = \int W_M\,\mathrm{d}A, \tag{1.2}$$

and

$$V_M = \int (M_1^2 - M_1 M_2 + M_2^2)^{\frac{1}{4}}\,\mathrm{d}A = \int W_M\,\mathrm{d}A, \tag{1.3}$$

where M_1 and M_2 are the principal bending moments. In what follows, we shall call the integrands W_T and W_M the specific cost functions since these express the material volume, or more generally "cost" of a plate referred to unit area of the middle plane. The problem so formulated was treated in several papers. Hopkins and Prager [1] considered a simply supported circular plate and obtained the solution for the Tresca material. The alternative solution for the Mises material was considered by Freiberger and Tekinalp [2], who derived the condition of analytical extremum; it requires the constant mean rate of energy dissipation $\bar{D}$ per unit area of the middle surface. Optimal design of solid annular plates was considered by Mróz [4]; it was shown in [4] that the condition $\bar{D}$ = const. corresponds to a local minimum only for states represented by corners of the Tresca hexagon. A similar problem for sandwich annular plates was treated by Megarefs [10, 11]. Sheu and Prager [12] sought the optimal solution for annular sandwich plates assuming piecewise constant thickness. The general criteria of optimum design were discussed by Drucker and Shield [3] and Mróz [5, 6]. A sufficient criterion for minimum volume is that the rate of energy dissipation $\bar{D}$ should be constant on the free surface that may be varied and be smaller outside of this surface than in the inerior of the body. However, this criterion does not apply to solid plates since the rate of energy dissipation increases with the distance measured along the normal to the middle plane; thus it is greater in the exterior of the plate than in its interior and the solution for $\bar{D}$ = const. may also correspond to a local maximum. This situation was encountered in [7] where the total cost of materials in reinforced slabs was to be minimized. Only for some stress states can the solution correspond to a local minimum. For the Tresca yield condition these states are represented by corners of the yield hexagon and for the Mises yield condition these are defined by the inequalities

$$-0{\cdot}725 \leq M_1/M_2 \leq 1{\cdot}725, \qquad -0{\cdot}725 \leq M_2/M_1 \leq 1{\cdot}725, \tag{1.4}$$

which follow from the condition of convexity of the specific cost function with respect to M_1 and M_2.

The character of optimal solutions for frames was recently discussed by Megarefs and Hodge [8] both for convex and concave cost functions. When the specific cost function is a concave function of a limit bending moment, $W = cM^{\alpha}$, $\alpha < 1$, it was shown that the absolute minimum may be non-analytical and does not correspond to constant mean rate of energy dissipation. A theory of optimal design for convex cost functions was presented by Marçal and Prager [9]. The present case, however, is not embraced by this theory since W_T and W_M as defined by (1.1) and (1.2) are not convex for all values of M_1 and M_2.

To investigate the character of absolute minimum solutions, we consider two examples: an infinite cantilever plate and a circular plate, both uniformly loaded. In addition to a design satisfying the condition $\bar{D}$ = const., an alternative design is considered for which a thin plate of sectionally constant thickness is reinforced by ribs. For the latter design the volume of plate can be smaller than that corresponding to the local minimum solution provided sufficiently high ribs are allowed. By passing to an infinite number of ribs, the volume of the plate tends to zero when the ribs become infinitely high.† Besides the theoretical result, the present analysis exhibits quantitatively the effect of reinforcing by ribs on limit load and economy of design. We shall also modify formulation of the problem of optimal design by introducing the notion of available space in which the solid plate carrying prescribed loads should be located.

It may be expected that similar effects occur for other criteria of optimum design. For instance, when maximum stiffness of an elastic structure is assumed as the design criterion, a theorem analogous to that for plastic structures has been proved [6]: for an absolute minimum to occur, the specific elastic energy must be constant on the free surface subjected to variation and be a decreasing function when moving in the exterior of that surface. Thus for solid plates the latter condition cannot be satisfied. This situation is similar to that encountered in the plastic design of solid plates.

† After preparation of this paper the authors became aware of the note by Brotchie [13] who demonstrated that the volume of an annular plate may tend to zero when it is reinforced by ribs of infinite thickness. It should be noted here that ribs of infinite thickness occur also in the design satisfying the condition $\bar{D}$ = const. Thus for sandwich plates such "design constraints" have been exhibited by Megarefs [10, 11] and investigated by Sheu and Prager [12] for plates with piecewise varying thickness; they also occur in solid plates designed for the condition $\bar{D}$ = const. [4]. Whereas such ribs occur due to necessity of satisfying boundary conditions, in the present paper these are introduced in order to achieve a better design than that satisfying the condition $\bar{D}$ = const.

2. A CANTILEVER PLATE

Consider an infinite plate of width b, built-in at one edge $y = b$ and subjected to uniform distributed transverse load q (Fig. 1b). Throughout the following analysis the Tresca yield condition will be used. The optimal design satisfying the condition $\bar{D} = \text{const.}$ is obtained for the stress state $M_x = M_{xy} = 0$, $M_y = -\frac{1}{2}qy^2$, from which the variable thickness $2h$ is obtained by setting $|M_y| = M_0 = \sigma_0 h^2$. Thus

$$2h = \left(\frac{2q}{\sigma_0}\right)^{\frac{1}{2}} y, \tag{2.1}$$

(Fig. 1a). The corresponding rate of curvature $\dot{\varkappa}_y$ is obtained from the condition of constant mean rate of energy dissipation:

$$\bar{D} = \frac{M_y \dot{\varkappa}_y}{2h} = \alpha, \qquad \dot{\varkappa}_y = \frac{C}{y}, \tag{2.2}$$

where α and C are positive constants.

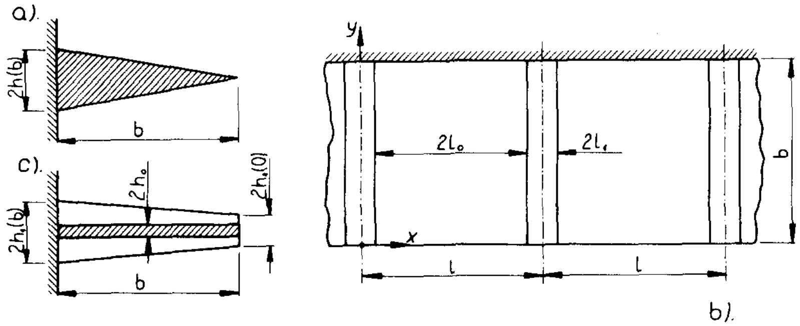

FIG. 1. A cantilever plate uniformly loaded; (a) stationary-volume design, (b) and (c) rib-reinforced plate.

Consider now the alternative design shown in Figs. 1b, c. The plate of constant thickness $2h_0$ is reinforced by ribs located at the distance l with linearly varying height $2h_1$. Let us introduce the non-dimensional geometric parameters

$$\frac{b}{l} = \varphi, \qquad \frac{l_0}{l} = \eta, \qquad \frac{l_1}{l} = \tfrac{1}{2}(1-2\eta). \tag{2.3}$$

In order to determine the thickness of plate and ribs, we shall apply both the static and the kinematic approach. Construct first a statically admissible stress field by assuming that the bending moments within the plate are carried only in the x-direction and neglecting the effect of the built-in edge. Thus the plate carries the transverse load in the same way as a set of beams that are parallel to the x-axis and clamped at their junctions with ribs. A maximum value of the bending moment is reached in the mid-section $x = b/2\varphi$ and in the end sections; assuming this value to be equal to the limit bending moment, we have $M_0 = \frac{1}{4}qb(\eta/\varphi)^2 = \sigma_0 h^2$, and

$$2h_0 = A\frac{\eta}{\varphi}, \qquad A = \left(\frac{q}{\sigma_0}\right)^{\frac{1}{2}} b. \tag{2.4}$$

The ribs are treated as plates, loaded uniformly by the pressure q; the principal moment directions are assumed to coincide with the x- and y-axes. The equilibrium equation

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2} = -q \tag{2.5}$$

is solved assuming that $M_x = M_x(x)$, $M_y = M_y(y)$ and the shear forces Q_x, Q_y are linear functions of x and y, respectively. We find

$$\begin{aligned} M_x &= q\eta\left[\frac{x^2}{1-2\eta}-\frac{1-\eta}{4}\left(\frac{b}{\varphi}\right)^2\right] = M_1, \\ M_y &= -\frac{qy^2}{2(1-2\eta)} = M_2. \end{aligned} \tag{2.6}$$

Since both principal moments are negative, the rib thickness is determined from the condition

$$M_0(x, y) = \sup[|M_1(x)|, |M_2(y)|]. \tag{2.7}$$

It turns out that for $y = 0$ the maximum value of M_x defines the thickness whereas at the clamped edge $y = b$ the thickness is defined by the moment M_y. Accordingly

$$2h_1(0) = A\frac{1}{\varphi}[\eta(1-\eta)]^{\frac{1}{2}}, \qquad 2h_1(b) = A\left(\frac{1}{1-2\eta}\right)^{\frac{1}{2}}. \tag{2.8}$$

Assuming linearly varying thickness of the rib between the values given by (2.8) (which circumscribes the rib profile calculated from (2.6)), we find the dimensionless volume of the typical plate portion contained within the strip of width l and fixed length b

$$\bar{V} = \frac{V}{Ab^2} = \frac{1}{\varphi^2}\left\{\varphi\left(\frac{1-2\eta}{2}\right)^{\frac{1}{2}}+\frac{1-2\eta}{2}[\eta(1-\eta)]^{\frac{1}{2}}+2\eta^2\right\}. \tag{2.9}$$

The formula (2.9) corresponds to upper bound of plate volume since the design is based upon the statically admissible stress field.

To obtain a lower bound, consider the failure mechanism shown in Fig. 2. The yield line along the built-in edge defines the failure mechanism of the whole plate whereas the yield lines AB, BD, AD, DC, BF, AE define the failure mechanism of the plate portion between rigid ribs. Equating the rate of work of external forces to the rate of dissipation along the yield lines, we find by superimposing the two mechanisms that

$$\frac{1}{2\varphi}qb^2\dot{w}_1+\frac{1}{3\varphi}qb^2\eta(3-\zeta_1)\dot{w}_2 = \frac{2}{\varphi}\left(M_0\eta+M_0'\frac{1-2\eta}{2}\right)\dot{w}_1+4M_0\left(\frac{\varphi}{\eta}+\frac{\eta}{\zeta_1\varphi}\right)\dot{w}_2, \tag{2.10}$$

where $\dot{w}_1$, $\dot{w}_2$ are shown in Fig. 2, and M_0, M_0' are the limit bending moments of the plate and the rib, respectively. From (2.10), we obtain

$$M_0 = \frac{qb^2}{12}\eta^2\zeta_1^2\frac{3-\zeta_1}{\zeta_1\varphi^2+\eta^2}, \qquad M_0' = \left(\frac{qb^2}{2}-2M_0\eta\right)(1-2\eta)^{-1}, \tag{2.11}$$

and

$$2h_0 = A\eta\left[\frac{\zeta_1}{3}\left(\frac{3-\zeta_1}{\zeta_1\varphi^2+\eta^2}\right)\right]^{\frac{1}{2}}. \tag{2.12}$$

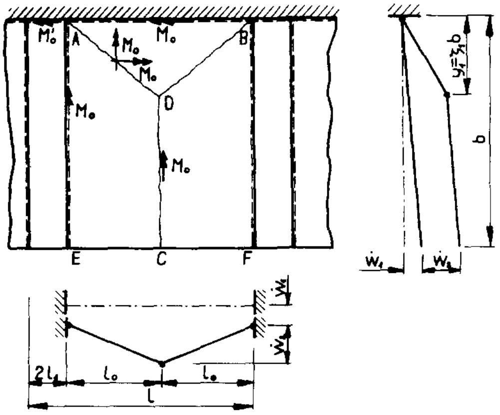

FIG. 2. Failure mechanism of cantilever plate.

The quantity ζ_1 should be determined from the extremum condition for q which yields

$$\zeta_1 = \left(\frac{\eta}{\varphi}\right)^2\left\{\left[1+3\left(\frac{\varphi}{\eta}\right)^2\right]^{\frac{1}{2}}-1\right\}. \tag{2.13}$$

The rib thickness at the built-in edge is obtained from the second equation (2.11)

$$2h_1(b) = \mathrm{A}\left\{\frac{2}{1-2\eta}\left[1-\frac{\eta^3\zeta_1}{3}\left(\frac{3-\zeta_1}{\zeta_1\varphi^2+\eta^2}\right)\right]\right\}^{\frac{1}{2}}; \tag{2.14}$$

it varies linearly along the y-axis, attaining the plate thickness $2h_0$ for $y = 0$. The non-dimensional volume of the typical portion of the plate of length l equals

$$\bar{V} = (\tfrac{1}{2}+\eta)\frac{\eta}{\varphi}\left[\frac{\zeta_1}{3}\left(\frac{3-\zeta_1}{\zeta_1\varphi^2+\eta^2}\right)\right]^{\frac{1}{2}}+\frac{1}{\varphi}\left\{(\tfrac{1}{2}-\eta)\left[1-\frac{\eta^3\zeta_1}{3}\left(\frac{3-\zeta_1}{\zeta_1\varphi^2+\eta^2}\right)\right]\right\}^{\frac{1}{2}}. \tag{2.15}$$

Figure 3 presents the variation of $\bar{V}$ in dependence on both η and φ. Note that, for $\eta = 0$, we obtain the volume of the plate satisfying the extremum condition (Fig. 1a). It is seen that for sufficiently large φ ($\varphi > 2{\cdot}0$), all designs correspond to smaller volumes than that for $\eta = 0$. When $\eta \to 0{\cdot}5$ and $\varphi \to \infty$, which corresponds to infinitely densely distributed ribs of zero width and infinite height, the plate volume tends to zero. This, of course, is only a theoretical result because the thin plate theory ceases to be valid for high ribs and the problem of lateral stability becomes important. It nevertheless indicates that the problem of optimal design considered above may not be correctly formulated. Note, furthermore, that for $\varphi > 0{\cdot}8$ the upper and lower bounds are very close; for smaller values of φ, the effect of the built-in edge should be accounted for in constructing the statically admissible stress field. Figure 3 presents also the ratio β_1 of the maximum thickness $2h_1(b)$ of the rib-reinforced plate calculated from the static solution to the maximum thickness $2h(b)$ of the plate of Fig. 1a. We see that β_1 monotonically increases with η and all plates of smaller volume than that for $\eta = 0$ are higher at the built-in edge. If the available space for plate is prescribed in the form of a strip $z = \pm t$, symmetrically on both sides of the middle plane, from Fig. 3 a proper design can be selected. When $2t > 2h(b)$, this design will correspond to a rib-reinforced plate and the solution (2.4) constitutes a limit case when $2t = 2h(b)$.

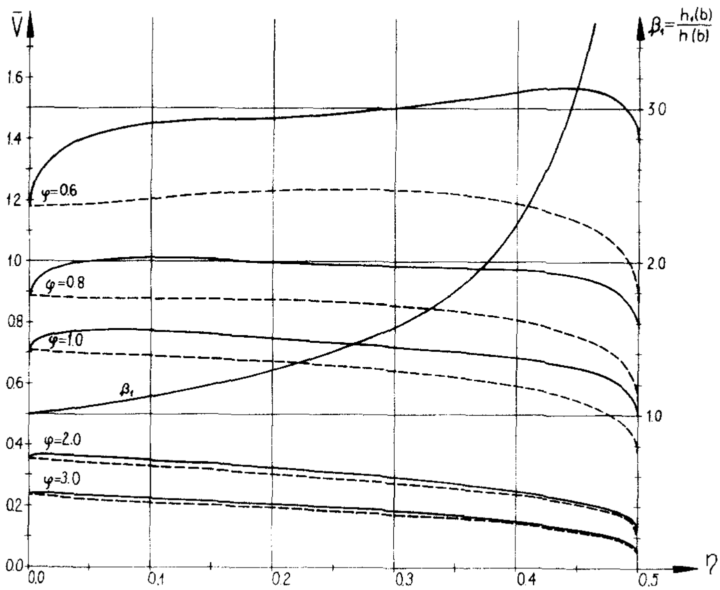

FIG. 3. Volume of plate portion of width l in function of η and φ, and the thickness ratio of two designs (curve β_1).

3. A SIMPLY SUPPORTED CIRCULAR PLATE

Consider the other case for which a complete solution satisfying the condition for local extremum exists: a circular plate, simply supported and subjected to the uniform transverse load q. For the Tresca yield condition, the stress state of the local minimum solution corresponds to a corner of the yield hexagon for which $M_r = M_\varphi = M_0 = \sigma_0 h^2$; the thickness and volume are given by

$$2h_0 = \left[\frac{q}{\sigma_0}(R^2 - r^2)\right]^{\frac{1}{2}}, \qquad V_0 = 0{\cdot}816\, V_c, \tag{3.1}$$

where M_r, M_φ are radial and circumferential bending moments, V_0 is the plate volume of the design (3.1) and V_c is the volume of the plate of constant thickness carrying the same limit load. This solution was given by Hopkins and Prager [1]. Later it was shown in [2] that (3.1) corresponds to local minimum whereas there is another solution for which the condition $\bar{D}$ = const. is satisfied and it corresponds to a local maximum of volume.

Now, we shall consider other designs and demonstrate that smaller volumes can be achieved in the class of plates reinforced by circumferential ribs. Similarly as previously, ribs are treated as plates of different thickness and complete solutions are sought for the Tresca yield condition.

Let the plate be divided into n annular regions $0 = r_0 < r_1 < r_2 \ldots < r_n = R$, each of constant thickness $2h_1, 2h_2, \ldots, 2h_n$ with the corresponding limit moments $M_0^{(1)}, M_0^{(2)}, \ldots, M_0^{(n)}$. Let there be $2h_1 < 2h_2 > 2h_3 < \ldots > 2h_{k-1} < 2h_k > 2h_{k+1}$ where $k = 0, 2, 4, \ldots$, Fig. 4. When $n = k_{n+1}$, by passing $r_{n-1} \rightarrow r_n = R$, the thickness of the outermost annulus tends to zero and the static field within the plate is the same as for the case $n = k_n$. On the other hand, when $n = k_n$ and $r_{n-1} \rightarrow r_n = R$, the thickness

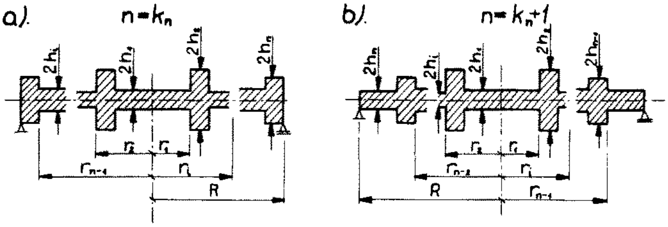

FIG. 4. Circular plate reinforced by circumferential ribs.

of the outermost annulus tends to infinity. We therefore consider the case $n = k_{n+1}$ and by setting $r_{n-1} \to R$, the case $n = k_n$ is obtained; the transition from the case $n = k_n$ to $n = k_{n-1}$, however, is not possible.

Since the annuli $r_k < r < r_{k+1}$ are regarded as being built-in at the interfaces, the following boundary conditions must be imposed

$$r = r_k\colon M_r(r_k) = M_0^{(k+1)}, \qquad r = r_{k+1}\colon M_r(r_{k+1}) = -M_0^{(k+1)}. \tag{3.2}$$

Thus within the interval $\langle r_{i-1}, r_i \rangle$ the radial moment should vanish at some radius $r = \rho_i$. The stress profile for the whole plate is shown in Fig. 5.

As is seen from Fig. 5a, for a typical annulus $\langle r_k, r_{k+1} \rangle$ the state of stress is represented by $A_{k+1}B_{k+1}$ within the region $\langle r_k, \rho_{k+1} \rangle$ and by $B_{k+1}C_{k+1}$ within the region $\langle \rho_{k+1}, r_{k+1} \rangle$ Similarly, within a thicker annulus $\langle r_{k+1}, r_{k+2} \rangle$ we have the state $C'_{k+2}B_{k+2}$ within the region $\langle r_{k+1}, \rho_{k+2} \rangle$ and $B_{k+2}A'_{k+2}$, $A'_{k+2}A''_{k+2}$ for $\rho_{k+2} < r < r_{k+2}$. The positive radial moment within the region $\langle r_{k+1}, r_{k+2} \rangle$ attains a maximum for $r = \rho'_{k+2}$ and drops to the value $M_0^{(k+2)}$ for $r = r_{k+2}$. For a narrow ring, it is possible that $\rho'_{k+2} = r_{k+2}$.

Integrating the equilibrium equation

$$\frac{\mathrm{d}}{\mathrm{d}r}(rM_r) - M_\varphi = -\frac{qr^2}{2} \tag{3.3}$$

and satisfying the yield condition for respective regions, we obtain the stress field throughout the whole plate. Let us introduce the relations

$$\gamma_i = \rho_i/R, \quad \gamma'_i = \rho'_i/R, \quad \delta_i = r_i/R, \quad \delta = r/R; \qquad i = 0, 1, 2, \ldots, n. \tag{3.4}$$

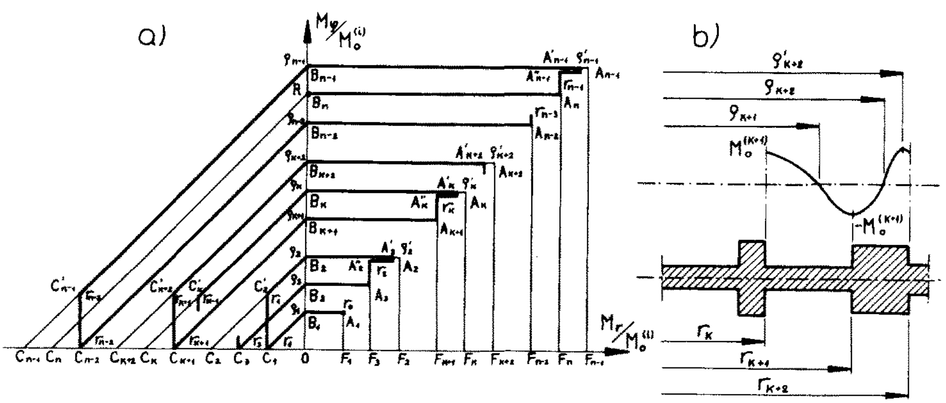

FIG. 5. Stress profile for the plate of Fig. 4.

The following formulae defining the stress field within the typical region $\delta_k^+ \le \delta \le \delta_{k+1}^-$ and in the outermost ring are obtained

$$\delta_k^+ \le \delta \le \gamma_{k+1}: \quad M_r = M_0^{(k+1)} - \frac{qR^2}{6}\delta^2\left[1-\left(\frac{\delta_k}{\delta}\right)^3\right], \quad M_\varphi = M_0^{(k+1)},$$

$$M_0^{(k+1)} = \frac{qR^2}{6}\gamma_{k+1}^2\left[1-\left(\frac{\delta_k}{\gamma_{k+1}}\right)^3\right];$$

$$\gamma_{k+1} \le \delta \le \delta_{k+1}^-: M_r = M_0^{(k+1)}\left(\ln\frac{\delta}{\delta_{k+1}}-1\right) - \frac{qR^2}{6}(\delta^2-\delta_{k+1}^2), \quad M_\varphi = M_r + M_0^{(k+1)},$$

$$M_0^{(k+1)} = \frac{qR^2}{4}(\delta_{k+1}^2-\gamma_{k+1}^2)\left(1-\ln\frac{\gamma_{k+1}}{\delta_{k+1}}\right)^{-1};$$

$$\delta_{k+1}^+ \le \delta \le \gamma_{k+2}: M_r = M_0^{(k+2)}\ln\frac{\delta}{\delta_{k+1}} - M_0^{(k+1)} - \frac{qR^2}{4}(\delta^2-\delta_{k+1}^2), \quad M_\varphi = M_r + M_0^{(k+2)},$$

$$M_0^{(k+2)} = \left[M_0^{(k+1)} - \frac{qR^2}{4}(\delta_{k+1}^2-\gamma_{k+2}^2)\right]\left(\ln\frac{\gamma_{k+2}}{\delta_{k+1}}\right)^{-1}; \tag{3.5}$$

$$\gamma_{k+2} \le \delta \le \delta_{k+2}^-: M_r = M_0^{(k+3)}\frac{\delta_{k+2}}{\delta} + M_0^{(k+2)}\left(1-\frac{\delta_{k+2}}{\delta}\right) - \frac{qR^2}{6}\delta^2\left[1-\left(\frac{\delta_{k+2}}{\delta}\right)^3\right],$$

$$M_0^{(k+2)} = M_0^{(k+3)} + \frac{qR^2}{6}\delta_{k+2}^2\left[1-\left(\frac{\gamma_{k+2}}{\delta_{k+2}}\right)^3\right]\left(1-\frac{\gamma_{k+2}}{\delta_{k+2}}\right)^{-1},$$

$$M_\varphi = M_0^{(k+2)};$$

$$\delta_{n-1}^+ \le \delta \le 1: \quad M_r = M_0^{(n)}\left(1-\frac{1}{\delta}\right) - \frac{qR^2}{6}\delta^2\left(1-\frac{1}{\delta^3}\right), \quad M_\varphi = M_0^{(n)},$$

$$M_0^{(n)} = \frac{qR^2}{6}(1-\delta_{n-1}^3).$$

From (3.5) we obtain the equation determining γ_i

$$1-\left(\frac{\delta_k}{\gamma_{k+1}}\right)^3 = \frac{3}{2}\left[\left(\frac{\delta_{k+1}}{\gamma_{k+1}}\right)^2-1\right]\left(1-\ln\frac{\gamma_{k+1}}{\delta_{k+1}}\right)^{-1}, \tag{3.6}$$

$$\left\{\gamma_{k+1}^2\left[1-\left(\frac{\delta_k}{\gamma_{k+1}}\right)^3\right]+\frac{3}{2}(\gamma_{k+2}^2-\delta_{k+1}^2)\right\}\left(\ln\frac{\gamma_{k+2}}{\delta_{k+1}}\right)^{-1}$$

$$= \left\{\gamma_{k+3}^2\left[1-\left(\frac{\delta_{k+2}}{\gamma_{k+3}}\right)^3\right]+\delta_{k+2}^2\left[1-\left(\frac{\gamma_{k+2}}{\delta_{k+2}}\right)^3\right]\right\}\left(1-\frac{\gamma_{k+2}}{\delta_{k+2}}\right)^{-1}.$$

The curvature rates are determined from the flow law for the sides A_iB_i, and B_iC_i of the Tresca hexagon. We have

$$\dot\lambda_i \ge 0, \quad \dot k_i = 0 \quad \text{on} \quad A_iB_i \quad \text{and} \quad \dot\lambda_i \ge 0, \quad \dot k_i \le 0, \quad \dot\lambda_i + \dot k_i = 0 \quad \text{on} \quad B_iC_i, \tag{3.7}$$

where

$$\dot\lambda_i = -\frac{1}{\delta}\frac{d\dot w_i}{d\delta}, \qquad \dot k_i = -\frac{d^2\dot w_i}{d\delta^2}. \tag{3.8}$$

The rates of displacements are expressed in the form

$$\dot{w}_i = D_i\delta + E_i \quad \text{on} \quad A_iB_i \quad \text{and} \quad \dot{w}_i = F_i \ln\delta + G_i \quad \text{on} \quad B_iC_i, \tag{3.9}$$

where: D_i, E_i and F_i, G_i are constants.

Satisfying the continuity conditions of $\dot{w}_i$ at junctions between adjacent regions, we obtain

$$\dot{w} = \begin{cases} (\dot{w}_{k+1} - \dot{w}_k)\left[1 + \left(1 - \ln\dfrac{\gamma_{k+1}}{\delta_{k+1}} - \dfrac{\delta_k}{\gamma_{k+1}}\right)^{-1} \ln\dfrac{\gamma_{k+1}}{\delta_{k+1}}\right]\dfrac{\delta - \delta_k}{\gamma_{k+1} - \delta_k} + \dot{w}_k, & \langle \delta_k, \gamma_{k+1} \rangle; \\ (\dot{w}_{k+1} - \dot{w}_k)\left(1 - \ln\dfrac{\gamma_{k+1}}{\delta_{k+1}} - \dfrac{\delta_k}{\gamma_{k+1}}\right)^{-1} \ln\dfrac{\delta}{\delta_{k+1}} + \dot{w}_{k+1}, & \langle \gamma_{k+1}, \delta_{k+1} \rangle; \\ (\dot{w}_{k+2} - \dot{w}_{k+1})\left(\dfrac{\delta_{k+2}}{\gamma_{k+2}} + \ln\dfrac{\gamma_{k+2}}{\delta_{k+1}} - 1\right)^{-1} \ln\dfrac{\delta}{\delta_{k+1}} + \dot{w}_{k+1}, & \langle \delta_{k+1}, \gamma_{k+2} \rangle; \\ (\dot{w}_{k+2} - \dot{w}_{k+1})\left(\dfrac{\delta_{k+2}}{\gamma_{k+2}} + \ln\dfrac{\gamma_{k+2}}{\delta_{k+1}} - 1\right)^{-1} \dfrac{\delta - \delta_{k+2}}{\gamma_{k+2}} + \dot{w}_{k+2}, & \langle \gamma_{k+2}, \delta_{k+2} \rangle; \\ \dot{w}_{n-1}(1-\delta)(1-\delta_{n-1})^{-1}, & \langle \delta_{n-1}, 1 \rangle. \end{cases} \tag{3.10}$$

It can be checked that the curvature rates satisfy the inequalities (3.7). There are hinge circles at $\delta = \delta_i$ where $\mathrm{d}\dot{w}_i/\mathrm{d}\delta$ changes in a discontinuous manner.

Our task is to select such values of δ_i which correspond to minimum of volume. In what follows, we shall discuss several particular cases. Obviously, the simplest case corresponds to a plate of constant thickness. Denoting its thickness by $2h_c$, the limit bending moment by M_0^c and the volume by V_c, we have

$$M_0^c = \frac{qR^2}{6}, \qquad 2h_c = \left(\frac{2q}{3\sigma_0}\right)^{\frac{1}{2}} R, \qquad V_c = \pi\left(\frac{2q}{3\sigma_0}\right)^{\frac{1}{2}} R^3. \tag{3.11}$$

For a plate composed of annuli of different thickness, we can write

$$2h_i = \beta_i\left(\frac{2q}{3\sigma_0}\right)^{\frac{1}{2}} R, \qquad V = \alpha\pi\left(\frac{2q}{3\sigma_0}\right)^{\frac{1}{2}} R^3, \tag{3.12}$$

where $\beta_i = h_i/h_c$ $(i = 1, 2, \ldots, n)$ and $\alpha = V/V_c$ are the thickness and volume coefficients for the plate considered. These coefficients characterize both the geometry of the plate and its volume relative to the plate of constant thickness.

i. *Plate with the exterior reinforcing ring*

In this case we have $\delta_1 = r_1/R$, $\delta_2 = r_2/R = 1$, $\gamma_1 = \rho_1/R$, $\gamma_2 = \rho_2/R$, $\gamma_2' = \rho_2'/R$ and $M_r(\gamma_1) = M_r(\gamma_2) = 0$, $M_r(\gamma_2') = \max$. The equations defining limit bending moments $M_0^{(1)}$, $M_0^{(2)}$ and γ_1, γ_2 are obtained from (3.5) and (3.6) upon setting $n = 3$ and $\delta_2 \to \delta_3 = 1$. We have

$$1 - \ln\frac{\gamma_1}{\delta_1} = \frac{3}{2}\left[\left(\frac{\delta_1^2}{\gamma_1}\right) - 1\right], \quad [\gamma_1^2 - \tfrac{3}{2}(\delta_1^2 + \gamma_2^2)]\left(\ln\frac{\gamma_2}{\delta_1}\right)^{-1} = 1 + \gamma_2 + \gamma_2^2. \tag{3.13}$$

From the first equation we obtain $\gamma_1 = 0{\cdot}730\,\delta_1$; the second equation is solved for several values of δ_1 and the results are presented in Fig. 6. For $\delta_1 > 0{\cdot}715$, the coefficient $\gamma_2 = 1$ and the point A_2' on the stress profile (Fig. 6) passes onto the M_φ-axis.

The limit bending moments are given by

$$M_0^{(1)} = \frac{qR^2}{6}\gamma_1^2 = b_0 h_1^2;$$
$$M_0^{(2)} = \frac{qR^2}{6}\left\{\begin{array}{ll}(1+\gamma_2+\gamma_2^2) & 0 \le \delta_1 \le 0{\cdot}715; \\ [\gamma_2^2-\tfrac{3}{2}(\delta_1^2-1)]\left(\ln\dfrac{1}{\delta}\right)^{-1} & 0{\cdot}715 \le \delta_1 \le 1.\end{array}\right\} = b_0 h_2^2, \tag{3.14}$$

Fig. 6 presents the thickness and volume coefficients β_1, β_2 and α. It is seen that for $\delta_1 > 0{\cdot}92$ the plate volume becomes smaller than that of plate of constant thickness and tends to a limiting value $\alpha = 0{\cdot}730$ for $\delta_1 \to 1$. The limiting case corresponds to the infinitely high rib of zero width. It should be noted that Hopkins and Prager [1] considered the plate of stepwise varying thickness; for $\delta_1 = 0{\cdot}8$ it was obtained $\alpha = 0{\cdot}93$. A better design in our case can be obtained for $\delta > 0{\cdot}97$, if the thickness coefficient $\beta_2 = 4{\cdot}4$ is admitted.

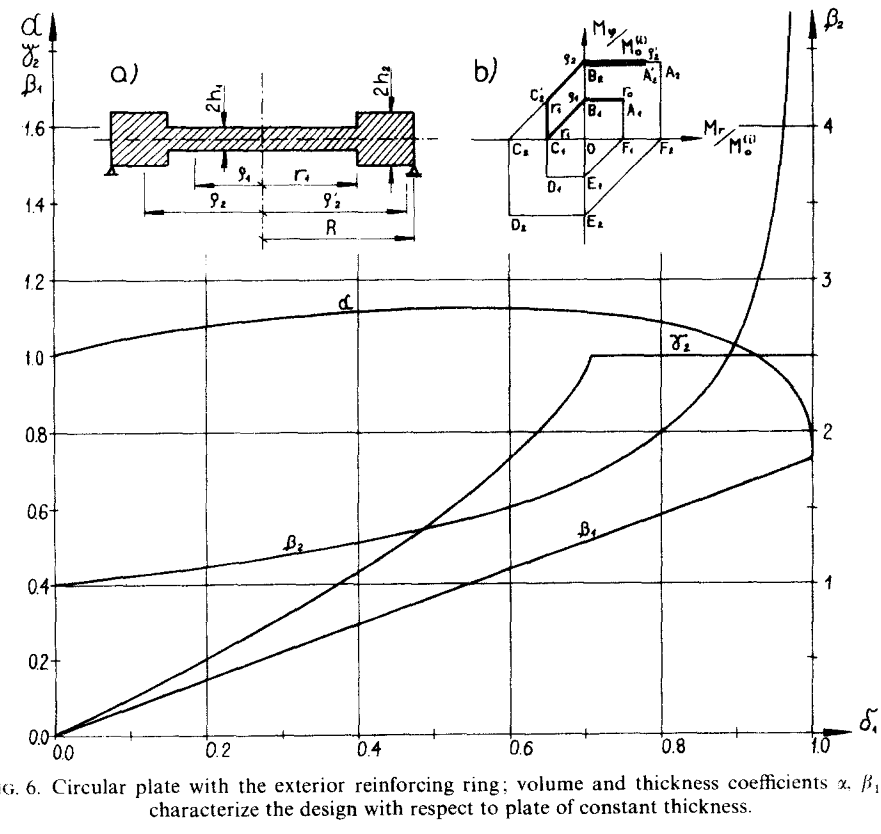

FIG. 6. Circular plate with the exterior reinforcing ring; volume and thickness coefficients α, β_1, β_2 characterize the design with respect to plate of constant thickness.

ii. *Plate with the interior reinforcing ring*

In this case there are two design parameters $\delta_1 = r_1/R$, $\delta_2 = r_2/R$ ($\delta_3 = 1$). The values of γ_1, γ_2 and of bending moments are obtained from the set (3.5), (3.6) by setting $n = 3$. Similarly as previously, we have $\gamma_1 = 0{\cdot}730\,\delta_1$ and the value of γ_2 is given by the equation

$$[\gamma_1^2+\tfrac{3}{2}(\gamma_2^2-\delta_1^2)]\left(\ln\frac{\gamma_2}{\delta_1}\right)^{-1} = \left\{\delta_2^2\left[1-\delta_2-\left(\frac{\gamma_2}{\delta_2}\right)^3\right]+1\right\}\left(1-\frac{\gamma_2}{\delta_2}\right)^{-1}. \tag{3.15}$$

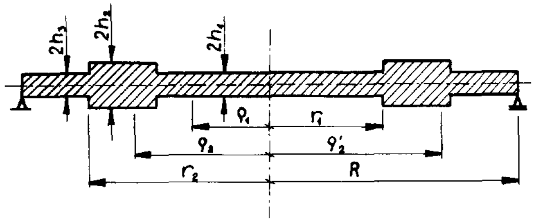

FIG. 7. Plate with the interior reinforcing ring.

The corresponding bending moments are given by

$$M_0^{(1)} = \frac{qR^2}{6}\gamma_1^2; \tag{3.16}$$

$$M_0^{(2)} = \frac{qR^2}{6}\begin{cases}\left\{\delta_2^2\left[1-\delta_2-\left(\dfrac{\gamma_2}{\delta_2}\right)^3\right]+1\right\}\left(1-\dfrac{\gamma_2}{\delta_2}\right)^{-1}, & \text{for } \delta_2 \neq 1 \text{ or } \delta_1 \le 0{\cdot}715 \text{ when } \delta_2 \to 1;\\[2mm] [\gamma_1^2+\tfrac{3}{2}(1-\delta_1^2)]\left(\ln\dfrac{1}{\delta_1}\right)^{-1}, & 0{\cdot}715 < \delta_1 \le 1 \text{ when } \delta_2 \to 1;\end{cases} \tag{3.17}$$

and the thickness coefficients β_1, β_2, β_3 are calculated from (3.16). The volume coefficient equals

$$\alpha = \delta_1^2(\beta_1-\beta_2)+\delta_2^2(\beta_2-\beta_3)+\beta_3. \tag{3.18}$$

The values of β_2 and α are given in Table 1. When $\delta_2 = 1$, we obtain the previously considered case of the plate reinforced by exterior ring. Setting $\delta_1 = 0$, the case of the plate with thickness increasing toward the center is obtained. It is seen from Table 1 that the volume coefficient α attains the smallest value $\alpha = 0{\cdot}625$ when $\delta_2 = \delta_1 = 0{\cdot}8$. This is theoretically the smallest volume which can be attained when the plate is reinforced by the interior rib of infinite thickness.

iii. *Plate reinforced by two ribs*

Because numerical calculations become lengthy in this case, we consider only the limiting case when $\delta_2 = \delta_1$, $\delta_3 = 1$. From the general formulae (3.5)–(3.6) it is found that the minimum value of the volume coefficient $\alpha = 0{\cdot}547$ is attained when $\delta_2 = \delta_1 = 0{\cdot}75$.

Table 1

δ_2 \ δ_1	1·0	0·9	0·8	0·7	0·6	0·5	0·3	0·1
1·0	$\to\infty$* 0·730†							
0·9	2·608 1·028	$\to\infty$ 0·631						
0·8	1·987 1·089	2·723 0·936	$\to\infty$ 0·625					
0·77			4·805 0·811					
0·75			3·715 0·848					
0·7	1·696 1·115	2·026 0·998	2·727 0·911	$\to\infty$ 0·661				
0·6	1·505 1·121	1·710 1·026	2·007 0·971	2·650 0·916	$\to\infty$ 0·724			
0·5	1·376 1·123	1·514 1·038	1·680 0·998	1·931 0·968	2·510 0·934	$\to\infty$ 0·793		
0·3	1·188 1·101	1·265 1·030	1·334 1·005	1·406 0·996	1·499 0·991	1·659 0·986	$\to\infty$ 0·917	
0·1	1·054 1·004	1·103 0·982	1·135 0·967	1·157 0·970	1·172 0·977	1·186 0·987	1·262 0·999	$\to\infty$ 0·989
0	1·000 1·000	1·040 0·941	1·062 0·931	1·071 0·938	1·070 0·952	1·061 0·967	1·031 0·990	1·005 0·9996

* β_2.
† α.

4. ALTERNATIVE FORMULATION OF THE PROBLEM

As is seen from the previous analysis, the volume of a solid plate carrying prescribed loading depends on the maximum height which is admitted in design. This implies an alternative formulation of the optimal design problem. Instead of looking for the variable height, we can assume that this is prescribed and the plate is reinforced by ribs in one or two perpendicular directions; the height of ribs $2h_2$ is equal to the prescribed plate thickness, Fig. 8a. Several types of design can now be considered. For instance, it can be assumed that the central sheet thickness $2h_1$ and the rib height $2h_2$ are given and the width of ribs in both directions is to be determined as well as their layout within the plate. The problem would become similar to that for plates reinforced by rods distributed along the lines of principal bending moments [7]. Here, we shall not pursue this problem but shall discuss a simpler case assuming that the circular plate is reinforced by densely distributed circumferential ribs of constant height $2h_2$ and the central sheet has variable thickness $2h_1$. Thus the two design variables l_1 and $2h_1$ should be determined. We shall treat the plate as arthotropic and the limit bending moments are defined by the relations, cf. Fig. 8

$$M_r = b_0 h_2^2 \varphi, \qquad M_\varphi = b_0 h_2^2[\eta + \varphi^2(1-\eta)], \tag{4.1}$$

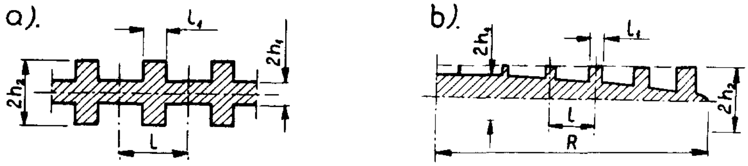

Fig. 8. Circular plate of constant thickness $2h_2$ with constant (a) and variable central sheet thickness $2h_1$ (b).

where $l_1/l = \eta$, $h_1/h_2 = \varphi$. Denoting $m_r = M_r/b_0h_2^2$, $m_\varphi = M_\varphi/b_0h_2^2$, the plate volume can be expressed as follows

$$V = 2h_2 \int \frac{m_r^{\frac{1}{2}} + m_\varphi}{1 + m_r^{\frac{1}{2}}} r \, \mathrm{d}r. \tag{4.2}$$

A static solution satisfying the local extremum condition of (4.2) can be obtained from the equilibrium (3.3) and the Euler's equation, the latter yielding

$$r\frac{\mathrm{d}m_r}{\mathrm{d}r} - 2m_r^{\frac{1}{2}} - 2m_r - m_\varphi + 1 = 0. \tag{4.3}$$

However, it can be checked that this solution corresponds to local volume maximum and should be rejected. Since the minimum may be non-analytical, it can only be numerically evaluated by considering various sets of statically admissible stress states. Assume, for instance, the following set depending on one parameter m

$$M_r = \tfrac{1}{16}q(3+m)(R^2 - r^2), \qquad M_\varphi = \tfrac{1}{16}q[R^2(3+m) - r^2(1+3m)]. \tag{4.4}$$

Let $h_2 = nh_0$, where $h_0 = (q/\sigma_0)^{\frac{1}{2}}R$ is the maximum thickness of the design (3.1). The plate volume corresponding to the static field (4.4) can be calculated from (4.2)

$$\alpha = \frac{V}{V_0} = 3\left\{\frac{7-3m}{6(3+m)^{\frac{1}{2}}} + \frac{n(1-m)}{3+m} - \frac{4n^2(1-m)}{(3+m)^{\frac{1}{2}}} + \frac{2n(1-m)}{3+m}\left(\frac{4n^2}{3+m} - 1\right) \ln\left[1 + \frac{(3+m)^{\frac{1}{2}}}{2n}\right]\right\} \tag{4.5}$$

where V_0 is the volume of the design (3.1). If we obtain $\alpha_0 < 1$, sufficiently large value of n should be assumed. For instance, when $n = 2$, $m = -2$, we have $\alpha_0 = 0{\cdot}974$. More economical designs than (3.1) can thus be obtained if the prescribed thickness is approximately twice as much as that of (3.1). Figure 8b presents the design for $n = 2$, $m = -2$.

5. CONCLUDING REMARKS

Solution satisfying the condition of constant rate of plastic work on lateral surfaces of solid plates may not correspond to a local minimum; moreover, they do not correspond to an absolute minimum of volume since theoretically a zero volume solution can be obtained. The examples presented in this paper illustrate quantitatively the effect of single and densely distributed ribs on economy of design. The problem of optimal design of solid plates (or shells) should thus be modified by limiting the maximum plate thickness or prescribing the region in which the plate is to be located.

It may be noted that similar peculiarities should be encountered when considering other types of optimal design, for instance requiring maximum stiffness or maximum buckling loads of elastic plates.

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REFERENCES

[1] H. G. HOPKINS and W. PRAGER, Limits of economy of materials in plates. *J. appl. Mech.* **22**, 372–374 (1955).
[2] W. FREIBERGER and B. TEKINALP, Minimum-weight design of circular plates. *J. Mech. Phys. Solids* **4**, 294–299 (1956).
[3] D. C. DRUCKER and R. T. SHIELD, Bounds on minimum-weight design. *Q. appl. Math.* **15**, 269–281 (1957).
[4] Z. MRÓZ, Limit analysis and minimum-weight design of annular plates. *Rozpr. Inż.* **4**, 605–625 (1958). (In Polish.)
[5] Z. MRÓZ, On a problem of minimum-weight design. *Q. appl. Math.* **19**, 127–135 (1961).
[6] Z. MRÓZ, Limit analysis of plastic structures subject to boundary variations. *Archwm Mech. Stosow.* **15**, 63–76 (1963).
[7] Z. MRÓZ, On the optimum design of reinforced slabs. *Acta mech.* **3**, 34–55 (1967).
[8] G. J. MEGAREFS and P. G. HODGE, Singular cases in the optimum design of frames. *Q. appl. Math.* **21**, 91–103 (1963).
[9] P. V. MARÇAL and W. PRAGER, A method of optimal design. *J. Méc.* **3**, 509–530 (1964).
[10] G. J. MEGAREFS, Method of minimal design of axisymmetric plates. *J. Engng Mech. Div. Am. Soc. civ. Engrs* **92**, No. EM6, 79–99 (1966).
[11] G. J. MEGAREFS, Minimal design of sandwich axisymmetric plates. *J. Engng Mech. Div. Am. Soc. civ. Engrs* **93**, No. EM6, 245–269 (1967).
[12] C. Y. SHEU and W. PRAGER, Optimal plastic design of circular and annular sandwich plates with piecewise constant cross section. *J. Mech. Phys. Solids* (1968).
[13] J. F. BROTCHIE, Discussion to the paper by G. J. Megarefs, *J. Engng Mech. Div. Am. Soc. civ. Engrs* **93**, 173–175 (1967).

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Абстракт—Задача об оптимальном проектировании сплошных пластин сводится к определению переменной толщины, соответствующей минимуму обьема при постоянной предельной нагрузке. До сих пор, условие постоянной скорости диссипации на внешних поверхностях пластины, обычно применялось при решении такой задачи. Но это условие соответствует только экстремуму обьема, которое вообще не представляет собой локального минимум, тем более не может представлять абсолютного минимума. Построены два примера проектирования консольной полосы и круглой пластины, где показано, что существует много решений, соответствующих меньшему обьему чем при выполнении условия постоянной скорости диссипации. Представлена другая формулировка задачи об оптимальном проектировании сплошных пластин.